BOUNDED CONFIDENCE WITH REJECTION: THE INFINITE POPULATION LIMIT WITH PERFECT UNIFORM INITIAL DENSITY

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Abstract – Huet and Deffuant (2007) propose a new opinion dynamics model based on the bounded confidence principles, with a rejection mechanism. The simulations of the agent-based model show that this new model leads generally to fewer clusters than the classical bounded confidence model. We build an aggregated model of this agent-based model (ABM), considering the limit case of an infinite population, in order to better understand the clustering process in the ABM. The dynamics of the aggregated model, starting from a perfectly uniform distribution of opinions, and the dynamics of the ABM are significantly different. However, the aggregated model provides relevant information about how the rejection mechanism and bounding the attitude space influence the clustering process. It firstly outlines that, with a perfectly symmetric density, the clustering process is almost impossible and that the only final stable state is one centred cluster. Secondly we notice that one centred cluster can be lead with the only rejection mechanism or with an unbounded attitude space.

Key words – opinion dynamics, multidimensional, attraction, reject, consensus, aggregated modelling

1 Introduction

In [2], we propose a new opinion dynamics models in its agent-based version. We add a rejection mechanism to the bounded confidence rules. We observe that the rejection leads generally to more consensus in the population than the only attraction process. In this paper, we develop an aggregated model of this new opinion dynamics agent-based model, and we study its behaviour.

Inspired from the dissonance theory [3], the model presented in [2] involves multidimensional attitudes based on the bounded confidence (BC) model of [4], with a rejection mechanism. An attitude toward an object is defined here as a psychological assessment of this object [5]. Considering two dimensional attitudes with an equal importance, our main assumption is that, when two individuals strongly disagree on attitude x_1, and are close on attitude x_2, they tend to solve the dissonance by shifting away on attitude x_2. As an example, [6] reports about students who, informed that their attitudes regarding a particular issue are close to the one of the Ku Klux Klan, decide to reinterpret this issue and to finally adopt an attitude further away from the one of the Ku Klux Klan. Practically, we follow the hypotheses of the BC models: people tend to be closer when they are sufficiently close. We add the rejection mechanism when people are close on one attitude and far on the other. Two models are usually identified as BC models: Deffuant [4] and Hegselmann-Krause [7] models. We start in this work from the communication regime used in [4]: agents meet in random pair wise encounters after which they influence each other. From simulations of the agent-based model, we know that in most cases, the rejection mechanism leads to fewer clusters than with the standard bounded confidence.

Our paper presents the aggregated modelling of the individual-based model. We follow the "double-modelling" approach, which consists in the study of an individual-based model, generally an agent-based model (ABM), by modelling this model. The expected benefit of such a work is to provide explanations of the collective effects observed in individual-based model simulations, through an aggregated view of the individual-based model behaviour [1]. In this paper we build the aggregated model on an approximation of infinite populations, and we write the differential equations ruling the evolution of the probability density of opinions. Practically, we have to discretise the opinion space to solve numerically these equations. The results exhibit a higher tendency to the consensus than in the ABM. This tendency is due to the perfect symmetry of the distribution, combined with the rejection rule.

The first part of the paper describes the ABM and its aggregated approximation. Then, we present the results of the aggregated model.

2 The ABM and its aggregated approximation

2.1 The ABM

The ABM considers a population of agents with bidimensionnal attitudes (or opinions), supposed initially uniformly distributed. To each attitude is associated an uncertainty, which is supposed constant u in this study, for sake of simplicity. We assume that agents meet by randomly chosen pairs. Suppose that individual A of attitudes a_1 and a_2, meets with individual B of attitudes b_1 and b_2. Let $(\delta a_1, \delta a_2)$ be the bidimensionnal vector of the changes of attitudes of A, because of B's influence (i.e, after the meeting a_1 becomes $a_1+\delta a_1$, a_2 becomes $a_2+\delta a_2$). This influence can be tuned with parameter μ. In the following, we only study $\mu = 0.5$.

Three cases occur:

1. If a_1 is close to b_1 and a_2 is close to b_2: $|a_1 - b_1| \leq u$ and $|a_2 - b_2| \leq u$, then, the rules of the BC model are applied on both dimensions, and A's attitudes are moved towards B's on both dimensions (attraction effect): $\delta a_1 = \mu(b_1 - a_1)$, $\delta a_2 = \mu(b_2 - a_2)$
2. If a_1 is far from b_1 and a_2 is far from b_2: $|a_1 - b_1| > u$ and $|a_2 - b_2| > u$, then, the rules of the BC model are also applied on both dimensions, and there is no influence: $\delta a_1 = 0$, $\delta a_2 = 0$
3. If the attitudes are close on one dimension (suppose: $|a_1 - b_1| \leq u$) and far on the other ($|a_2 - b_2| > u$), then a_1 moves away from b_1. The movement is the highest when these opinions are equal, and tends to zero linearly, when their difference approaches u.
   if $a_1 - b_1 < 0$ then : $\delta a_1 = -\mu(u - (b_1 - a_1))$, $\delta a_2 = 0$
   if $a_1 - b_1 \geq 0$ then : $\delta a_1 = \mu(u + (b_1 - a_1))$, $\delta a_2 = 0$

Figure 1 illustrates the different types of interactions. Note that we only get symmetrical interactions: if A attracts B, B attracts A; if A rejects B, B rejects A.

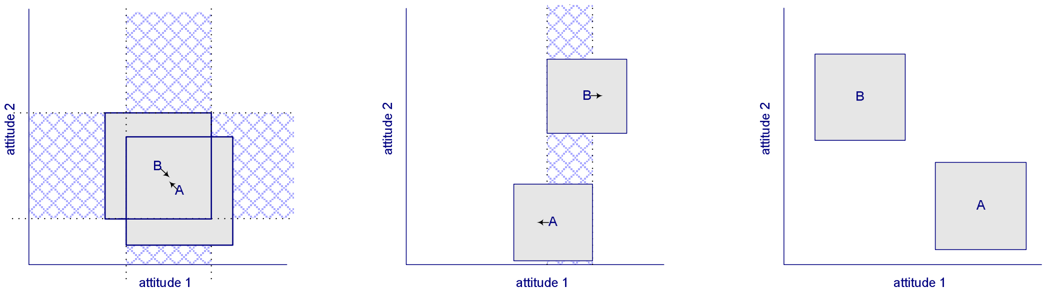

Fig. 1. *A* and *B* in a situation: of attraction on both dimensions ((left); of rejection on one dimension (on attitude 1 dimension here) and no influence on the other dimension (center); of no influence (right)

In [2], we suppose in addition that the attitudes are confined within the initial bounds of the distribution of attitudes (when the rejection pushes an attitude outside the bounds, we bring it back on the boundary). In these conditions, we observed that this model converges towards a set of metastable clusters. The number of these clusters is found to vary like $1/u$, whereas it is varies like $(1/u)^2$ in the standard bounded confidence model. We now elaborate an aggregate model of this ABM, in order to get a better understanding of this clustering process.

2.2 Aggregate model at the limit of infinite population

We consider the limit case of an infinite population, with an initially perfectly uniform distribution of attitudes on $[-M,+M] \times [-M,+M]$ with $M = 1$. To solve numerically the aggregated model, we discretise the compact $[-1,1] \times [-1,1]$, with regular grid of size m x m (typically, we take 100 x 100). On each point $g(i,j)=(-1+i/m,-1+j/m)$ of the grid, $\rho(i,j)$ represents the probability that an agent of the population has their attitudes in a square of center (i,j), and of size $1/m$. Moreover, we discretise the boundary of the domain, with values of i and j equal to 0 or $m+1$.

Distribution ρ is initially perfectly uniform and is null on the boundary, thus:

$$\begin{aligned} &\rho(i,j) = {}^{1}\!/_{m^2}\,, (i,j) \in \{1,..,m\} \times \{1,..,m\} \\ &\rho(i,j) = 0 \text{ if } \; i = 0 \text{ or } i = m+1 \text{ or } j = 0 \text{ or } j = m+1 \end{aligned} \tag{1}$$

The principle of the model dynamics is to compute the flows of distribution from one site (i,j) to any other site (k,l), and to sum them up to compute the distribution change (all sites are updated at the same time).

More precisely, for couple of attitude $g(i,j)$, we consider all the couples of attitude $g(k,l)$, and we compute the influence of $g(k,l)$ on $g(i,j)$. Let $(\delta a_1, \delta a_2)$ be the change on attitude dimension 1 and 2, computed with the rules presented above. Let ($\lfloor a \rfloor$ means the integer part of number a):

$$\delta i = \left\lfloor \frac{\delta a_1}{2M} m \right\rfloor, \text{ and } \delta j = \left\lfloor \frac{\delta a_2}{2M} m \right\rfloor \tag{2}$$

The probability of encounter between agents of site (i,j) and agents of site (k,l) is proportional to the product $\rho(i,j)\rho(k,l)$. Therefore, the global change of the distribution $d\rho$, due to the systematic encounters between all pairs of site is computed as follows:

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Computation of $d\rho$
For $(i,j)\in\{0,...,m+1\}\times\{0,...,m+1\}$ do:
    For $(k,l)\in\{0,...,m+1\}\times\{0,...,m+1\}$ do:
        If $\delta i=\left\lfloor \frac{\delta a_1}{2M} m \right\rfloor \neq 0$ or $\delta j=\left\lfloor \frac{\delta a_2}{2M} m \right\rfloor \neq 0$
            $d\rho(i+\delta i, j+\delta j) := d\rho(i+\delta i, j+\delta j)+\rho(i,j)\rho(k,l)$
            $d\rho(i,j) := d\rho(i,j)-\rho(i,j)\rho(k,l)$
        end if
    end for
end for
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Then, the global evolution of the probability density ρ is simulated numerically, as follows.

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Repeat:
    Compute $d\rho$
    For $(i,j)\in\{1,...,m\}\times\{1,...,m\}$ do: $\rho(i,j) := \rho(i,j)+d\rho(i,j)$
    Reset $d\rho$ to 0.
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3 The results

3.1 Low uncertainty

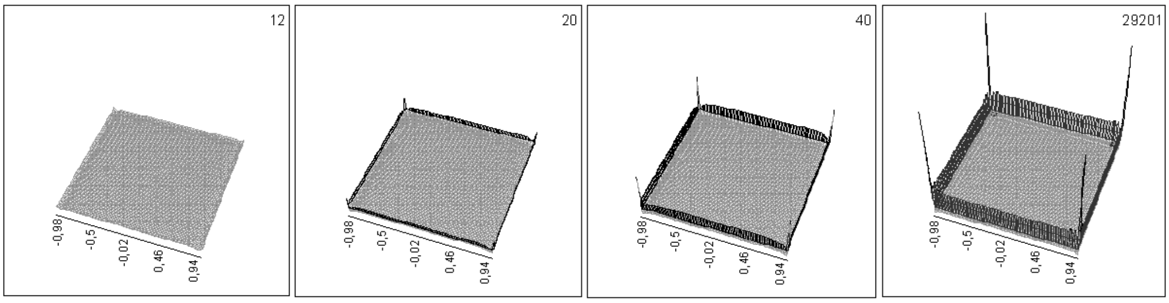

Fig. 2. Example of the effect on dynamics of the bounding of the attitude space for $u = 0.14$ and $M = 1$ for all *t (scale maximum 0.0011) No more movement at iteration 29201*

Figure 2 shows evolution of the probability density $\rho(i,j)$ for an uncertainty u equal to 0.14. Due to the rejection process, the density form peaks on the corners and the boundaries. However, we can not really say that we have clusters because the density on the corners is only 0.001, 0.0003 on the boundaries and 0.00009 in the center. The distribution is globally quite uniform. The increase of the corners stops at the iteration 29201 because the rejection leading the density on the corner, and the rejection leading the density out of the corner, compensate each other.

For low uncertainties, the rejection mechanism determines the dynamics. Indeed, the attraction influence on a given attitude is local: beyond a distance u, the attraction is null. The rejection influence is more global. Its scope depends on the size of the attitude space (equals to $2M$) and the uncertainty value. Its global intensity can be approximated at the first iteration where the density distribution is uniform by (*M*/*u*)-1. Thus, for a given *M,* the weaker *u*, the larger is the influence of the rejection on a given attitude compared with a constant local attraction influence. Thus, it will be more and more difficult for the attraction to form clusters when *u* decreases because of the global opposing rejection mechanism.

2.1 Average uncertainty

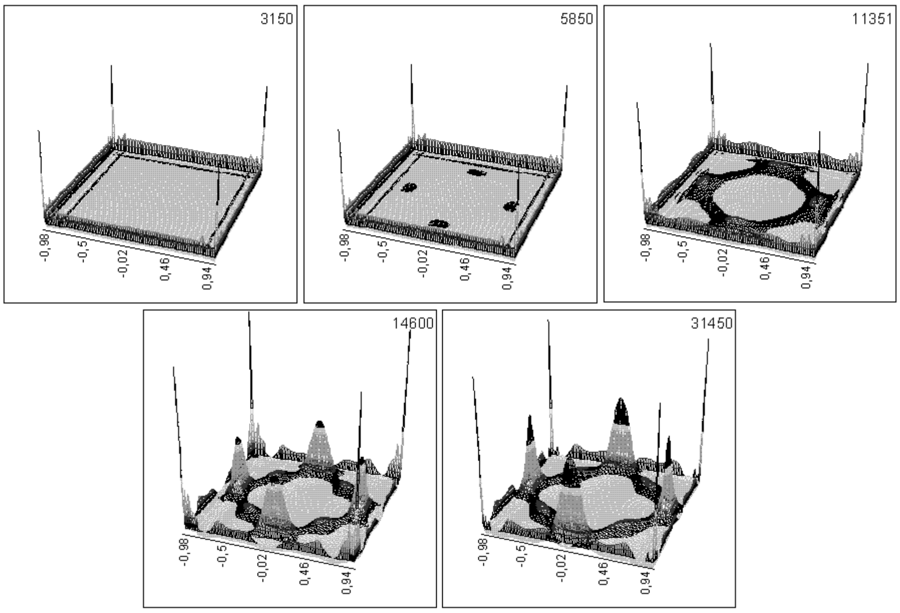

Fig. 3. Trajectory on time for $u = 0.19$ and $M = 1$ for all t (maximum of the z-axis for all diagrams 0.0025)
No more movement at iteration 29151

Figure 3 shows the trajectory for an uncertainty $u = 0.19$. At the beginning, like for a low uncertainty, we observe peaks in the corners. However the dynamics continues and creates four clusters in the center, opposed by couple on horizontal and vertical lines. Corner clusters and centered clusters evolve since they are in a metastable state. We have finally a complex equilibrium formed of interdependent small clusters. However, these peaks are very small: the maximum density is 0.0025.

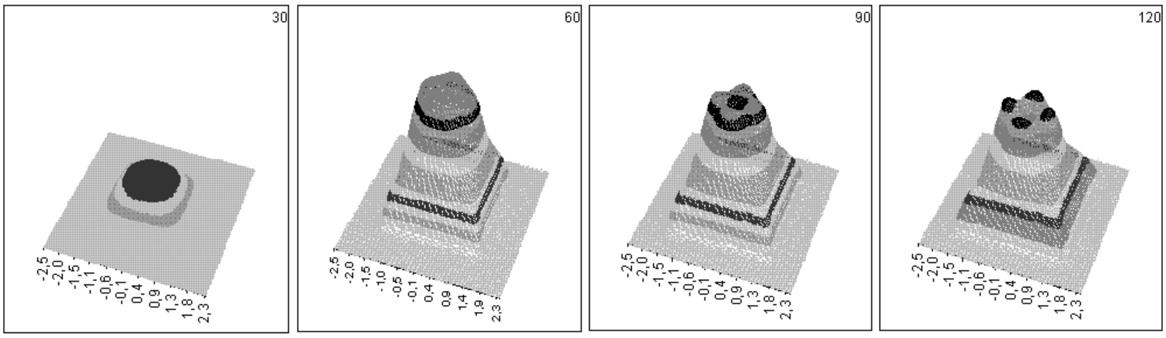

Fig. 3bis. "4 clusters" appearance for an unbounded attitude space for $u = 0.35$ and M = 1 at the iteration 0
(the maximum of the z-axis is 0.00036 for the four diagrams)

Figure 3bis shows how the "four clusters in the center" can be formed with the same dynamics applied on an unbounded attitude space for an uncertainty equal to 0.35. It allows us to conclude that, even if the dynamics of rejection into the corners interferes with the dynamics inside the attitude domain, it does not mainly explain the formation of these four clusters. Indeed, it occurs also with an unbounded attitude space where no peaks appear on the corners.

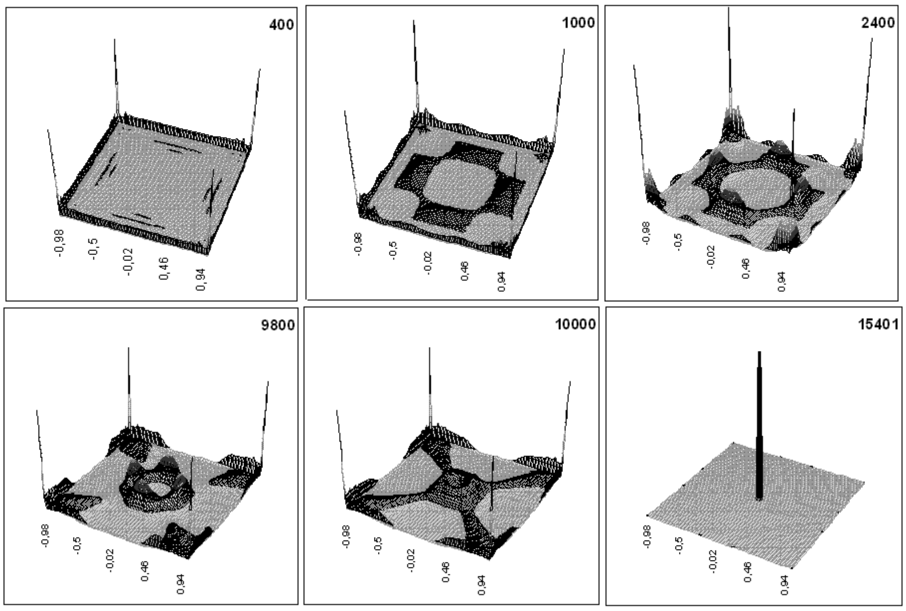

Fig. 4. Trajectory on time for $u = 0.25$ and $M = 1$ for all t (maximum of the z-axis for all diagrams 0.0035 for the fifth diagram)

Figure 4 shows a partially different evolution of the probability density for an uncertainty $u = 0.25$. Much of the trajectory is identical to the previous observed one. However the dynamics continue: the four central clusters concentrate themselves to the center and the corner densities join the center by the diagonals. It is really difficult to explain how the inside attitude domain dynamics and the boundary dynamics interfere to finally lead to only one centred cluster. However, as figures 4 bis and ter show, each dynamics, separated from the other, can be sufficient to obtain one final central cluster.

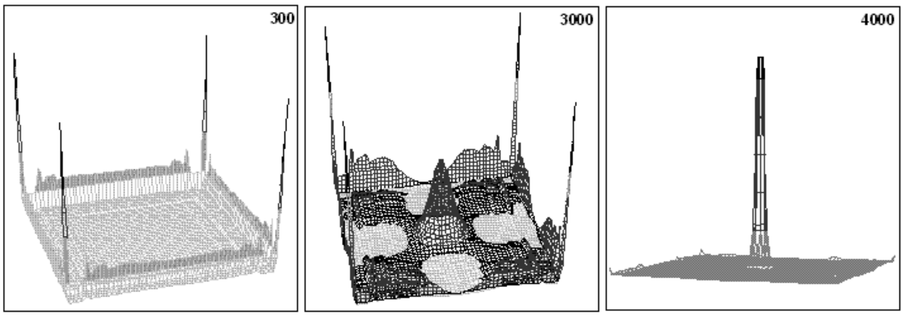

Fig. 4bis. Trajectory of $u = 0.4$ for an only rejection model on a bounded attitude space (max of the two first diagrams: 0.0036).

Indeed, on the figure 4bis, model includes only the rejection dynamics on a bounded attitude domain. The trajectory shows that the density is coming back from the corners to the centre by the diagonals to finally concentrate in the center. Indeed, corners reject each other. Due to the borders, the only way that these rejected flows can take is the diagonal lines going to the centre. As the centre is the only "symmetrical" stable position for a cluster, the density increases there. However, one can notice that this occurs for u = 0.4, not for u = 0.25 as in figure 4. Indeed, for u = 0.25, the model has the same final state as the one showed in figure 2. It proves that for such uncertainty values, border dynamics and inside attitude domain dynamics interfere.

The figure 4ter shows that for an unbounded attitude space, u = 0.3 and μ = 0.2, the model leads to only one centred cluster. It is the same for u = 0.2 and μ = 0.2. In this case, we see that bounding the attitude space is not determining to obtain only one centred cluster. However, we can notice that it is right for μ = 0.2, not for μ = 0.5 which is the μ value of the example shown by the figure 4. Thus, as already shown, for such uncertainty values, border dynamics and inside attitude domain dynamics interfere.

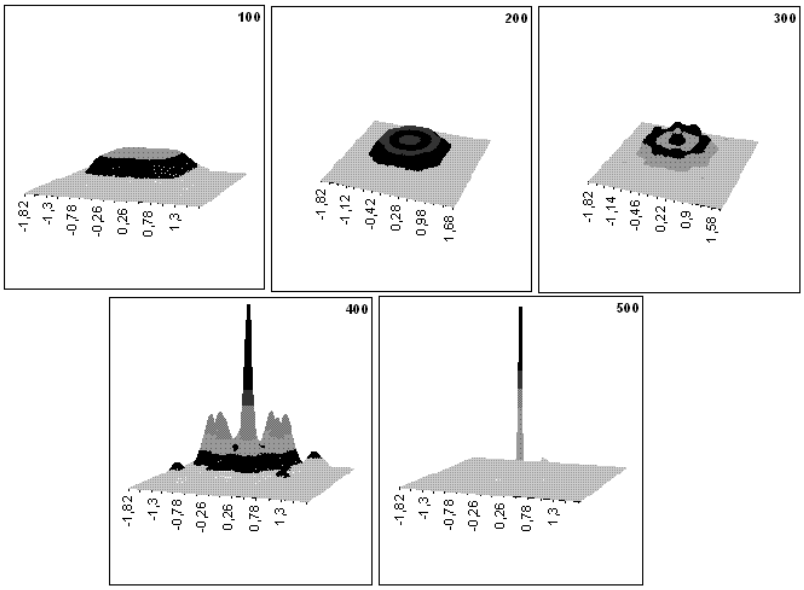

Fig 4.ter Trajectory of the model on an unbounded attitude space for u = 0.3, μ=0.2 and M = 1 at the beginning (maximum of the z-axis for the fifth diagram 0.006 except for the last diagram)

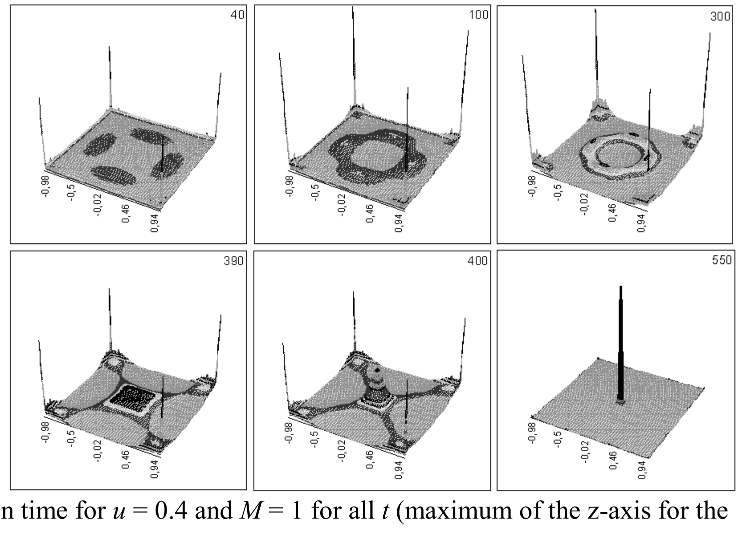

Fig. 5. Trajectory on time for u = 0.4 and M = 1 for all t (maximum of the z-axis for the fifth diagram 0.007)

Figure 5 shows that the trajectory remains globally the same for an uncertainty equal to 0.4. For this uncertainty values, both the border effect and the inside peaks lead to one final central cluster without being helped by the other one. Thus, the final state is reached very fast compared with the case $u = 0.25$.

Let's notice that results from the BC model study [8][9] with only attraction indicate that for uncertainties 0.25 or 0.4, the model leads to respectively 16 and 4 clusters. We can see there how rejection and boundary increase the consensus.

2.1 For high uncertainties

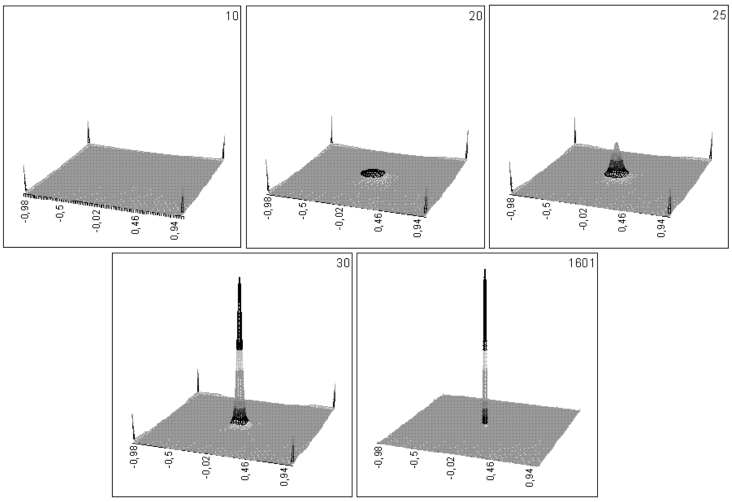

Fig. 6. Example of the effect on dynamics of the bounding of the attitude space for $u = 0.71$ and $M = 1$ for all t (maximum of the z-axis for all diagrams 0.04 except for the last one) No more movement at iteration 31450

As figure 6 shows, for high uncertainties, the model does not really appear different from the BC model with only an attraction mechanism. We just observe four minor clusters on the corners which are generally higher than the "minor clusters" of the BC models described by [8] [9] and resulting from the speed of the convergence to the centre.

3 Discussion and Conclusion

The aggregated model confirms the observations we made on individual based dynamics model: the rejection mechanism tends to increase the global consensus. Our main observations are:

- For high values of uncertainty, the aggregated model leads to one major central cluster.
- For low values of uncertainty, the aggregated model shows the difficulty of the dynamics to form clusters from a perfectly uniform distribution because rejection and attraction have antagonist effects. Thus, the local attraction needs sufficiently high local maxima of density distribution to create clusters. These local maxima can be obtained by the rejection flows around the borders of the initial attitude space. The border effect creates progressively local maxima of the density. It occurs on a band of width equal to u around the border. Then, the smaller u is, the smaller are these local maxima. Thus, the dynamics can create clusters if u is high enough to create high enough local maxima.

- For an intermediate range of u, we obtain a dynamic equilibrium where 4 small local maxima sustain each other.

Compared to the classical bounded confidence model, our attraction-rejection model leads more often to only one centred cluster. Indeed, for a $\mu = 0.5$ and an uncertainty going from approximately 0.25 to 0.54, while the bounded confidence model leads to 16 to 4 clusters [4][8][9], our model leads to one centred cluster.

Indeed, the centred major cluster is the only stable final state of the model. We know from the simulated results obtained with the agent-based model that a state can be stable only if you have at most one cluster on the same horizontal or vertical lines of width u. Indeed, if we suppose two symmetrical close clusters, attraction occurs and leads to one cluster; if we suppose two symmetrical clusters far from each other, rejection occurs (for $\delta = 0$). In our aggregated model, the density distribution is totally symmetric. In this case, if we have one cluster at a given attitude, we have at least another one on the same horizontal or the same vertical line of width u of the density matrix. The only exception is the centre of the matrix. It is the only symmetric attitude density with cluster which does not imply rejection.

Moreover, for this range of uncertainty value, we notice that the model can also lead one centred cluster if the attitude space is unbounded. Thus, to bound the attitude space is not determining to obtain one centred cluster. It also leads one centred cluster when it only considers a rejection mechanism between people on a bounded attitude space (i.e. no attraction). In the latter case, density peaks in the corners come back to the centre by the diagonal lines because of the boundary effect. Indeed, corners' groups reject each other in diagonal lines. The model also appears as very sensitive to the speed of rejection and attraction, the μ value, in this range.

However, this is a first study, and some other final states could exist. We have also to study more the impact of variousμ. We note that the aggregated model can lead to different final state that the agent based model. This is most probably due to the perfect symmetry of the initial distribution, which is an impossible situation in the ABM. Continuing the "double-modelling" approach [1] cited at the beginning of this paper, we plan to compare a non-symmetric version of the aggregated model to the agent-based model in order to better understand the specificity of the individual-based modelling.

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